Pergamon

Int. J. Solids Structures Vol. 32, No. 13, pp. 1893–1905, 1995

0020-7683/95 $9.50 + .00

0020–7683(94)00219–3

A PARTITIONED-MODELING APPROACH WITH MOVING JUMP CONDITIONS FOR LOCALIZATION

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(*Received* 25 *March* 1994; *in revised form* 26 *August* 1994)

Abstract—Although many investigations have been performed on localization problems, there still exist some pressing issues. Specifically, it is difficult to show in general well-posed governing equations. Based on the essential features of localization phenomena, a partitioned-modeling approach is proposed here via moving jump conditions for localization problems. By taking the initial point of localization as that point where the type of the governing differential equation changes, i.e. a hyperbolic to an elliptic type for dynamic problems and an elliptic to another elliptic type for static problems, a moving boundary between localized and non-localized deformation zones is defined through jump forms of conservation laws across the boundary. As a result, localization problems might be considered in the same category as shocks in fluids and solidification in heat transfer. To illustrate the proposed procedure, one-dimensional analytical solutions are given with an emphasis on the definition of boundary conditions and the experimental means to determine model parameters associated with localization. Future research is then discussed on an extension to general cases.

1. INTRODUCTION

Localization is manifested by degradation of material properties with localized large deformations, and this feature often results in formation and propagation of macrocracks through engineering structures. Due to the importance of localization phenomena in structural safety assessment, much research has been conducted to resolve experimental, theoretical and computational issues associated with localization problems, as reviewed by Chen and Schreyer (1994). Among the continuum models proposed are nonlocal plasticity and damage models, rate-dependent models, Cosserat continuum models and micromechanical models. To predict the essential feature of localization, several specific forms of nonlocal models have been suggested and include the use of gradients of strain (Chen and Schreyer, 1987; Mühlhaus and Aifantis, 1991; Triantafyllidis and Aifantis, 1986; Schreyer and Chen, 1986), imbricated continua (Bazant *et al.*, 1984), prescribed zones of localization (Pietruszczak and Mroz, 1981) and weighted integral averages of damage (Pijaudier-Cabot and Bazant, 1987). Recently, research on localization has been focused on resolving more efficiently different orientations and patterns of localization zones so that routine applications of localization analyses might be feasible (Chen and Schreyer, 1994; De Borst *et al.*, 1993; Neilsen and Schreyer, 1993; Pijaudier-Cabot and Benallal, 1993; Pietruszczak and Niu, 1993; Vardoulakis *et al.*, 1992; Zbib and Jubran, 1992). Although various promising analytical and numerical approaches exist in the current literature, there are still some pressing limitations that prohibit the successful prediction of localization phenomena in a general case. Among the continuum models proposed, for instance, the experimental means to identify model parameters and the physics behind boundary conditions are not well understood for higher-order models; mode-I failures cannot be properly regularized by the Cosserat approach; and the slow loading rates or the rate-independent limit generally can not be specified by rate-dependent models. More importantly, it is not easy to show in general well-posed governing equations as defined by John (1982).

In fact, the key component of various modeling approaches for localization is nothing but controlling the evolution of inhomogeneous interactions among material particles within a zone of finite dimension. In a macromechanical sense, however, the evolution process might be equally well characterized by the formation and propagation of a material surface discontinuity associated with a change of material properties. Based on the essential features of localization phenomena observed and on the previous work related to nonlocal models, a partitioned-modeling approach has been proposed with the introduction of moving boundaries for nonlocal plasticity and damage (Chen, 1993a,b; Chen and Wang, 1993). The basic ingredients of the approach are that different local constitutive models are used in different sub-domains with a moving boundary being defined between two sub-domains, and that the constitutive law governing the boundary is of an *ad hoc* nature based on localization mechanisms. As a result, simplified governing differential equations can be formulated in the partitioned domains for given boundary and initial conditions, and the evolution of localization can be represented by the moving boundary without invoking higher order terms.

To establish a sound mathematical foundation for the partitioned-modeling approach, an attempt is made in this paper to investigate the use of moving jump conditions in defining the moving boundary. By taking the initial point of localization as that point where the type of the governing equations changes, i.e. a hyperbolic to an elliptic type for dynamic problems and an elliptic to another elliptic type for static problems, a moving boundary between localized and non-localized deformation zones is defined through moving jump forms of conservation laws across the boundary. Jumps in density, velocity, strain and stress can be accommodated on this moving surface of discontinuity between the two zones. As a result, localization problems might be considered in the same category as shocks in fluids and solidification in heat transfer. In other words, those phase-transition problems, as characterized by the change in the type or coefficients of the governing equations (Chen and Clark, 1994), might be handled via a unified approach in terms of jump conditions. To illustrate the proposed partitioned-modeling procedure with the use of moving jump conditions, one-dimensional analytical solutions are given with an emphasis on the definition of boundary conditions and the experimental means to determine model parameters associated with localization. It appears that rate-independent local constitutive models can still hold in the post-limit regime within partitioned-domains, and the evolution of localization can be simulated by the moving boundary that is defined via a set of moving jump conditions.

2. A ONE-DIMENSIONAL ANALYTICAL APPROACH

An analytical method to solve a one-dimensional localization problem can be found under certain assumptions as elucidated next, although analytical solutions to localization problems might not be feasible in general due to the complexity of the mechanisms involved. Because of the novel nature of the partitioned-modeling approach with moving jump conditions, the analytical solutions presented provide a heuristic way to an extension to a general case. A numerical approach for three-dimensional problems might then be developed based on an analytical foundation. Since the essential feature of the proposed method is to employ a set of jump forms of conservation laws, basic concepts in this regard are first reviewed in the next subsection, and analytical solutions are then given and discussed in the remaining subsections.

2.1. *Jump forms of conservation laws*

For a tensile bar of length L as depicted in Fig. 1(a) with u being displacement in direction x, the conservation laws of mass and linear momentum in the spatial description (Achenbach, 1984) take the forms of

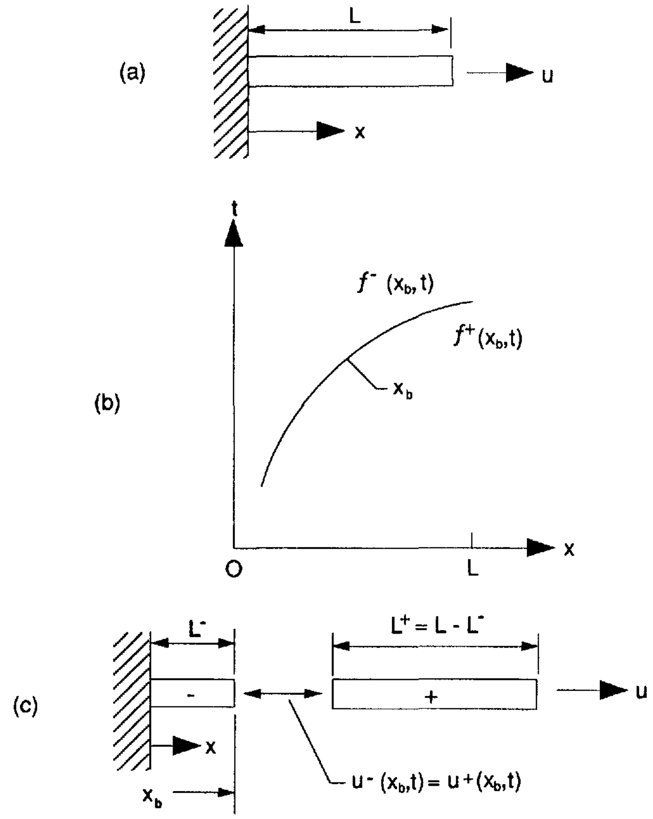

Fig. 1. (a) Bar analysed; (b) Jump across a moving boundary in the xt-plane; (c) Separation of the bar into two zones.

$$\rho_{,t}+(\rho v)_{,x} = 0 \tag{1}$$

and

$$-(\rho v)_{,t}+(\sigma-\rho v^2)_{,x} = 0 \tag{2}$$

respectively. In eqns (1) and (2), ρ denotes the mass density, v particle velocity along the x-axis, t time and σ normal stress. The jump forms in the xt-plane corresponding to eqns (1) and (2) can be found via a standard procedure (Chorin and Marsden, 1979) to be

$$-v_b\lfloor\rho\rfloor+\lfloor\rho v\rfloor = 0 \tag{3}$$

and

$$v_b\lfloor\rho v\rfloor+\lfloor\sigma-\rho v^2\rfloor = 0 \tag{4}$$

respectively. The symbol $\lfloor f\rfloor \equiv f^+(x_b, t)-f^-(x_b, t)$ represents the jump in the relevant field variables across a moving boundary denoted by x_b, as shown in Fig. 1(b), and v_b is the speed of the moving boundary in the xt-plane. These jump conditions place a restriction on the possible states that can exist on either side of the discontinuity moving with velocity v_b. To make use of the jump conditions, the entire bar is replaced with two zones partitioned by the moving boundary, as shown in Fig. 1(c), if localization occurs.

Because of the ease in obtaining analytical solutions, the small deformation assumption is employed below to derive the differential equations governing the zones to the left and right of the moving boundary. However, the jump conditions in eqns (3) and (4) are kept in a general form since the assumption that deformations within each zone are small does not preclude the possibility of large jumps in quantities between the zones.

2.2. *Analytical solutions for a dynamic case*

The equation of motion for a dynamic bar in the absence of body forces can be written as

$$\sigma_{,x} = \rho u_{,tt} \tag{5}$$

in the xt-plane, and the strain–displacement relation reads

$$\varepsilon = u_{,x} \tag{6}$$

with ε being normal strain. If the stress–strain relation is linearly elastic with Young's modulus E, the differential equation governing the wave traveling at the speed $v_e = \sqrt{E/\rho}$ along the bar is hyperbolic, and standard analytical solutions are available (Achenbach, 1984). The stress is doubled when the stress wave front reaches the rigid boundary at $x = 0$, and the localization occurs at that point if the doubled stress exceeds the limit strength. However, the evolution of localization can not be predicted by rate-independent local constitutive models due to the ill-posed resulting governing equations at the transition from the pre-limit to the post-limit regimes (Chen, 1993a).

To remedy the deficiency of local models, it is assumed here that the change of type of the equations initiates a moving boundary of discontinuity associated with a change of material properties. Thus, the entire problem domain can be divided by the moving boundary into two parts, as illustrated in Fig. 1(c), as long as the limit strength is reached at $x = 0$. Two kinds of local models are then used in the partitioned domains to the left and to the right of the moving boundary, respectively, which results in an elliptic equation inside the localization zone. In other words, the initial point of localization is defined as that point at which a hyperbolic equation is changed to be an elliptic one. If nothing is imposed, the problem is not solvable. If a regularity condition in time is imposed for the elliptic region through the moving boundary, a solution can be obtained. With the displacement continuity, $u^-(x_b, t) = u^+(x_b, t)$, imposed across the moving boundary, analytical field solutions to the governing equations can then be determined for the entire domain under given boundary and initial data, as described as follows.

For the domain to the right of the moving boundary, i.e., $x_b < x \leqslant L$, the original elastic relation

$$\sigma^+ = E\varepsilon^+ \tag{7}$$

still holds with the superscript + denoting the field variables outside the localization zone. Hence, the governing differential equation is still hyperbolic, namely,

$$\frac{E}{\rho^+} u^+_{,xx} - u^+_{,tt} = 0. \tag{8}$$

However, the boundary and initial data in the xt-plane now become

$$u^+(0, t) = 0 \quad t \leqslant t_L \tag{9a}$$

$$u^+(L, t) = at \tag{9b}$$

$$u^+\left(x, t_L + \frac{x}{v_b}\right) = u^+(x_b, t) \quad t > t_L \tag{9c}$$

$$u^+(x, 0) = 0 \tag{9d}$$

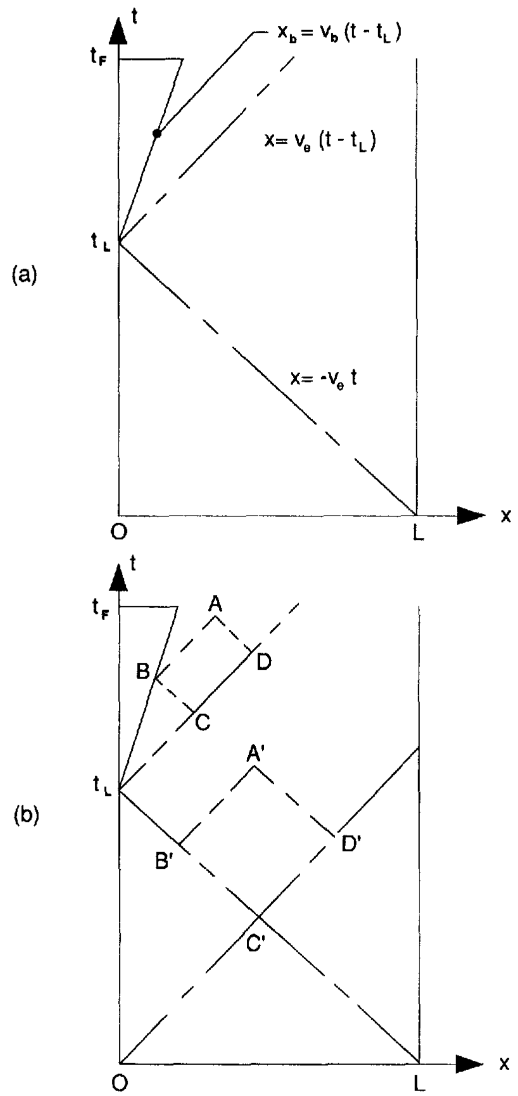

Fig. 2. (a) Solution domains partitioned by a moving boundary, $x_b = v_b(t-t_L)$, for a dynamic bar; (b) Parallelograms bounded by four characteristic curves as indicated by dashed lines.

$$u_{,t}^{+}(x,0) = 0 \tag{9e}$$

with $t_L = L/v_e$ being the time taken for the elastic wave front starting at $x = L$ at $t = 0$ to reach the point $x = 0$, and a a constant velocity prescribed at $x = L$. The assumption that v_b is constant has been made in eqn (9c) to obtain a closed-form solution. As can be seen from the above data and Fig. 2(a), the problem reduces back to elastic wave propagation with $\pm v_e$ being the slopes of characteristic curves, if eqn (9c) is deleted. Nevertheless, analytical solutions to eqn (8) under given data in eqn (9) still follow the standard forms for all the points to the right of the moving boundary $x_b = v_b(t-t_L)$ in the xt-plane. It should be noted that $v_b < v_e$ has been assumed in Fig. 2(a), which is consistent with fracture mechanics theory, and that x_b should not reach $x = L$ at failure.

For the domain to the left of the moving boundary, i.e. $0 \leqslant x < x_b$, a rate-independent local softening model is postulated as

$$\sigma^{-} = \sigma_L - \beta E(\varepsilon^{-} - \varepsilon_L) \quad \varepsilon^{-} \geqslant \varepsilon_L \tag{10}$$

where β is a softening parameter, and σ_L limit strength with $\varepsilon_L = \sigma_L/E$. The superscript $-$

denotes the field variables inside the localization zone. Equation (10) describes phenomenologically the degradation effect on the stress–strain relation, and holds as long as the boundary of discontinuity moves to the right monotonically, which implies no unloading occurs within the localization zone. Analytical solutions obtained by Bazant and Belytschko (1985) for local softening models also exclude the possibility of unloading. However, the evolution of localization can not be predicted with their approach because no jump in the mass density is allowed. As can be found from eqns (3) and (4), no jump in density results in no jumps in other field variables. Hence, a jump in density must be invoked if a local model is used to predict the moving boundary of discontinuity associated with the evolution of localization.

The use of eqns (5), (6) and (10) yields the differential equation governing the localization zone,

$$\frac{\beta E}{\rho^-} u^-_{,xx} + u^-_{,tt} = 0 \tag{11}$$

which is elliptic due to the softening of material properties. The following set of data is prescribed for eqn (11):

$$u^-(0, t) = 0 \quad t > t_L \tag{12a}$$

$$u^-\left(x, t_L + \frac{x}{v_b}\right) = u^-(x_b, t) \quad t > t_L \tag{12b}$$

$$u^-_{,x}(0, t_L) = \varepsilon_L \tag{12c}$$

$$u^-_{,x}(x, t_F) = \varepsilon_F \tag{12d}$$

in which a limit point is initiated at $x = 0$ with $\varepsilon = \varepsilon_L$. The physical interpretation of the parameter ε_F is that the load-carrying capacity of the bar is lost if a final average strain over the localization zone is reached for which continuum theory does not hold any more. As can be observed from Fig. 2(a), the above set of data is defined along a closed triangle, and eqn (12d) is imposed on the solution as a regularity condition.

Equation (11) with the boundary data in eqn (12) has a unique solution once v_b and $u^-(x_b, t)$ are determined. The solution can be expressed as

$$u^-(x, t) = c_0 + c_1 x + c_2(t - t_L) + c_3 x(t - t_L) \tag{13}$$

with c_0, c_1, c_2 and c_3 being the constants to be determined from given data. The use of eqns (12) and (13) then yields the displacement, velocity, strain and stress fields inside the localization zone:

$$u^-(x, t) = \varepsilon_L x + \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} x(t - t_L) \tag{14a}$$

$$v^-(x) = \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} x \tag{14b}$$

$$\varepsilon^-(t) = \varepsilon_L + \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L}(t - t_L) \tag{14c}$$

$$\sigma^-(t) = \sigma_L - \beta E \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L}(t - t_L) \tag{14d}$$

which must be related to the response outside the localization zone for the completeness of the physical problem. The response information can be transferred between the two parts of the bar through the displacement continuity at the moving boundary. In fact, the analytical solutions to eqn (8) under given data in eqn (9) are also incomplete without invoking the displacement continuity condition and jump conditions, as illustrated as follows.

Based on the theory of hyperbolic differential equations (John, 1982), it follows that for any parallelogram $A'B'C'D'$ in the xt-plane bounded by four characteristic curves as shown in Fig. 2(b), the sums of the solutions to eqn (8) in opposite vertices are equal, namely,

$$u(A')+u(C') = u(B')+u(D'). \tag{15}$$

Thus, for any point A in the xt-plane between the moving boundary $x_b = v_b(t-t_L)$ and the characteristic curve $x = v_e(t-t_L)$ originating at the point $(0, t_L)$, we have

$$u^+(A) = u^+(B)+u^-(D)-u^+(C). \tag{16}$$

Since the continuity condition requires $u^+(B) = u^-(B)$ and we can find along the line CD

$$u^+ = a\left\langle t+\frac{x-L}{v_e}\right\rangle \tag{17}$$

with $\langle f\rangle = (f+|f|)/2$, the solution $u^+(A)$ can be determined as long as the relations among the coordinates of the points A, B, C and D are known. As can be seen from Fig. 2(b), the response information from the localization zone only affects the solutions at the points between lines $x_b = v_b(t-t_L)$ and $x = v_e(t-t_L)$ originating at the point $(0, t_L)$, due to $v_b < v_e$.

Based on analytic geometry, the coordinates of the points B, C and D, as functions of the position (x, t) of point A, are given by

$$\begin{cases} x^B = v_b\dfrac{x-v_e(t-t_L)}{v_b-v_e} \\ t^B = \dfrac{x+v_bt_L-v_et}{v_b-v_e} \end{cases} \tag{18}$$

$$\begin{cases} x^C = \dfrac{(v_b+v_e)[x-v_e(t-t_L)]}{2(v_b-v_e)} \\ t^C = \dfrac{(v_b+v_e)(x-v_et)-v_et_L(v_e-3v_b)}{2v_e(v_b-v_e)} \end{cases} \tag{19}$$

and

$$\begin{cases} x^D = \dfrac{x+v_e(t-t_L)}{2} \\ t^D = \dfrac{x+v_e(t+t_L)}{2v_e} \end{cases} \tag{20}$$

respectively. With the use of eqns (14a), (16) and (17), it follows that outside the localization zone the displacement field is given by

$$u^+(x,t) = u^-(A) = \varepsilon_L x^B + \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} x^B (t^B - t_L) + a\left\langle t^D + \frac{x^D - L}{v_e}\right\rangle - a\left\langle t^C + \frac{x^C - L}{v_e}\right\rangle. \quad (21a)$$

After calculating derivatives of eqns (18)–(21a) with respect to x and t, respectively, corresponding velocity, strain and stress fields can be found to be

$$v^+(x,t) = -\frac{v_b(2a - \varepsilon_L v_e)}{v_e - v_b} + \frac{2v_b v_e(\varepsilon_F - \varepsilon_L)[v_e(t - t_L) - x]}{(t_F - t_L)(v_e - v_b)^2} \quad (21b)$$

$$\varepsilon^+(x,t) = \frac{2a - \varepsilon_L v_b}{v_e - v_b} - \frac{2v_b(\varepsilon_F - \varepsilon_L)[v_e(t - t_L) - x]}{(t_F - t_L)(v_e - v_b)^2} \quad (21c)$$

$$\sigma^+(x,t) = E\varepsilon^+(x,t). \quad (21d)$$

If v_b is determined, eqns (14) and (21) give complete analytical solutions to the field variables in the partitioned-domains of the dynamic bar, which describe the essential feature associated with localization. To find v_b and to explore the solution properties, the jump forms of conservation laws, eqns (3) and (4), are used next.

Because x is equal to $v_b(t - t_L)$ along the moving boundary, i.e. $x_b(t) = v_b(t - t_L)$, the values of velocity and stress at both sides of the boundary can be expressed, with the use of eqns (14) and (21), as functions of t only, namely,

$$v^-(x_b) = \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L} v_b(t - t_L) \quad (22a)$$

$$\sigma^-(x_b) = \sigma_L - \beta E \frac{\varepsilon_F - \varepsilon_L}{t_F - t_L}(t - t_L) \quad (22b)$$

and

$$v^+(x_b) = -\frac{v_b(2a - \varepsilon_L v_e)}{v_e - v_b} + \frac{2v_b v_e(\varepsilon_F - \varepsilon_L)(t - t_L)}{(t_F - t_L)(v_e - v_b)} \quad (23a)$$

$$\sigma^+(x_b) = E\left[\frac{2a - \varepsilon_L v_b}{v_e - v_b} - \frac{2v_b(\varepsilon_F - \varepsilon_L)(t - t_L)}{(t_F - t_L)(v_e - v_b)}\right] \quad (23b)$$

respectively. The use of eqns (3), (4), (22) and (23) then results in two equations involving two unknowns, $\rho^-(x_b)$ and v_b, with the assumption that $\rho^+(x_b)$ is equal to the original mass density ρ. After rearrangement of the terms, these two equations can be represented by

$$\rho^-(x_b) = \rho\frac{v^+(x_b) - v_b}{v^-(x_b) - v_b} \quad (24)$$

and

$$\sigma^-(x_b) = \sigma^+(x_b) - \rho[v^+(x_b) - v^-(x_b)][v^+(x_b) - v_b] \quad (25)$$

in which eqn (24) has been employed to derive eqn (25) from eqn (4) so that eqn (25) involves only one unknown v_b. Under the small deformation assumption, the products of small quantities can be omitted so that a linear approximation for v_b follows from eqn (25), namely,

$$v_b = (1-\beta)v_e \tag{26}$$

which is valid for $0.1 \leqslant \beta \leqslant 0.9$ with strain being much less than 1. Substituting eqn (26) into eqn (24) then yields the jump in mass density:

$$\lfloor \rho \rfloor = \rho\left\{1 - \frac{(t_F - t_L)[v_e(\varepsilon_L - \beta) - 2a] + 2v_e(\varepsilon_F - \varepsilon_L)(t - t_L)}{\beta v_e[(\varepsilon_F - \varepsilon_L)(t - t_L) - (t_F - t_L)]}\right\}. \tag{27}$$

Similarly, the jumps in velocity and stress can be obtained with the use of eqns (22), (23) and (26).

As can be seen from eqn (27), $\lfloor \rho \rfloor = 0$ at $t = t_L$, if $a = \varepsilon_L v_e/2$. It then follows from eqns (3) and (4) that the jumps in field variables all become zero, due to $\lfloor \rho \rfloor = 0$. In other words, no localization occurs if the condition, $a = \varepsilon_L v_e/2$, is prescribed at $x = L$, because the magnitude of the stress wave doubled at the rigid boundary, $\sigma = E(a/v_e)$, just reaches the limit strength. If the boundary condition is within the range of $(\varepsilon_L v_e/2) < a < \varepsilon_L v_e$, however, localization starts to evolve at $t = t_L$, which introduces the jumps in mass density, velocity and stress. For given a, these jumps at $t = t_F$ reach the values of

$$\lfloor \rho \rfloor = \left\{1 - \frac{2a + v_e(\beta - \varepsilon_L) - 2v_e(\varepsilon_F - \varepsilon_L)}{\beta v_e[1 - (\varepsilon_F - \varepsilon_L)]}\right\}\rho \tag{28a}$$

$$\lfloor v \rfloor = \frac{2(1-\beta)}{\beta}[a + v_e(\varepsilon_F - 2\varepsilon_L)] \tag{28b}$$

$$\lfloor \sigma \rfloor = E\left[\frac{2a + v_e(\beta\varepsilon_L + 3\varepsilon_L - 2\varepsilon_F)}{\beta v_e}\right]. \tag{28c}$$

As can be observed from eqn (28a), $\rho^+(x_b) < \rho^-(x_b)$ at $t = t_F$ because $\beta \gg \varepsilon$, although the jump in mass density is small under the small deformation assumption. The decrease of mass density inside the localization zone might be due to the increase of microcrack density associated with the evolution of localization. In addition, eqn (28c) indicates that the jump in stress is finite but might not be small, although the continuous deformations in two zones are small, respectively.

Since macro- and micro-experimental techniques available are unable to measure detailed variations of deformations inside a localization zone (Chen and Schreyer, 1994; Chen and Wang, 1993), the analytical solutions presented here might provide a simple way to identify model parameters. In fact, the partitioned-modeling approach with moving jump conditions can be verified through experiments if the average strain and dimension of a localization zone which is often very small are measurable. For the one-dimensional case under static loading, analytical solutions obtained for nonlocal softening models (Schreyer, 1990), which were verified by experimental data (van Mier, 1984; Shah and Gopalaratnam, 1985), provide an interesting example. To illustrate this, let us apply the proposed approach to a static bar, and see how the model parameters associated with localization can be determined via simple experiments.

2.3. *Analytical solutions for a static case*

As illustrated in Fig. 3, the static bar is defined on a closed rectangle in the xt-plane, with t parameterizing the static loading process. At $t < t_L$, a linear elastic stress–strain relation yields an elliptic equation of the type

$$u_{,xx} = 0 \tag{29}$$

which governs the whole bar. If the limit strength is reached at $t = t_L$ and $x = 0$, the loss of ellipticity is regularized by the initiation of a moving boundary, $x_b = v_b(t - t_L)$, across which

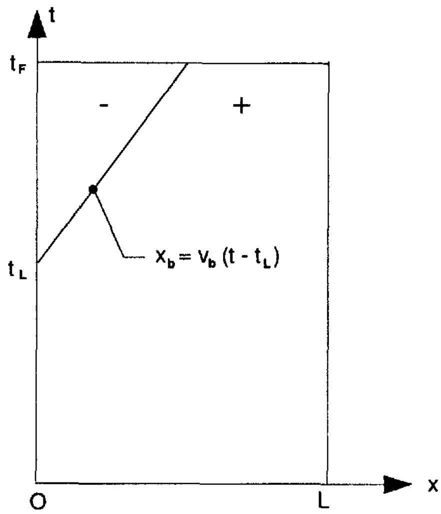

Fig. 3. Solution domains partitioned by a moving boundary for a static bar.

jumps in field variables occur with the evolution of localization. Two kinds of local models, eqns (7) and (10) are then employed in the partitioned domains to the left and to the right of the moving boundary, respectively. If the localization process is assumed to be of a dynamic nature, eqn (11) holds inside the softening zone with the data given by eqn (12). Outside the softening zone, eqn (29) is still valid for the following data:

$$u^+_{,x}(x, t_L) = \varepsilon_L \tag{30a}$$

$$u^+_{,x}(x, t_F) = \varepsilon_R \tag{30b}$$

$$u^+\left(x, t_L + \frac{x}{v_b}\right) = u^-(x_b) \quad t > t_L \tag{30c}$$

with ε_R denoting residual strain. As a result, two different elliptic equations with given data govern the whole bar after the limit point is reached. It should be noted that load or displacement can not be prescribed at $x = L$, because the post-limit response, including snap-back or snap-through, cannot be traced without using an evolving-localization constraint (Chen, 1993b; Chen and Schreyer, 1991). In fact, $u(L, t)$ should come out of the solution process, as shown next.

The displacement, velocity, strain and stress fields inside the localization zone are given by eqn (14) since the same governing differential equation and data are used here as in the dynamic case. The solution of eqn (29) for given data [eqn (30)] can be found by inspection to be

$$u^+(x, t) = \left[\varepsilon_L - \frac{\varepsilon_L - \varepsilon_R}{t_F - t_L}(t - t_L)\right]x + v_b \frac{\varepsilon_F - \varepsilon_R}{t_F - t_L}(t - t_L)^2 \tag{31a}$$

and corresponding velocity, strain and stress then take the forms of

$$v^+(x, t) = -\frac{\varepsilon_L - \varepsilon_R}{t_F - t_L}x + 2v_b \frac{\varepsilon_F - \varepsilon_R}{t_F - t_L}(t - t_L) \tag{31b}$$

$$\varepsilon^+(t) = \varepsilon_L - \frac{\varepsilon_L - \varepsilon_R}{t_F - t_L}(t - t_L) \tag{31c}$$

$$\sigma^+(t) = E\varepsilon^-(t). \tag{31d}$$

It follows from eqn (31a) that at $t = t_F$,

$$u^+(x, t_F) = \varepsilon_F v_b(t_F - t_L) \tag{32}$$

with $\varepsilon_R = 0$ and $v_b(t_F - t_L)$ being the final width of the localization zone, which implies that snap-back or snap-through would occur in the load-displacement space, i.e. $\varepsilon_L L \geqslant \varepsilon_F v_b(t_F - t_L)$, if the size of a localization zone is small compared to the whole bar. Hence, a control signal must be taken from the localization zone instead of the load or displacement level prescribed at $x = L$, if experimental data associated with the post-limit response are required.

If v_b is determined, eqns (14) and (31) provide complete analytical field solutions of a static bar in the post-limit regime. Similar to the dynamic case, the jump in linear momentum, eqn (25), is used here with eqns (14) and (31) at $x = v_b(t - t_L)$. Under the small deformation assumption, the same expression for v_b as in the dynamic case, eqn (26), is arrived at for the rate-independent models used in the partitioned-domains. With the use of eqns (24), (14) and (31), the jumps in mass density, velocity and stress can then be found to be

$$\lfloor \rho \rfloor = \left[1 - \frac{(t_F - t_L) - (t - t_L)(2\varepsilon_F - \varepsilon_L - \varepsilon_R)}{(t_F - t_L) - (t - t_L)(\varepsilon_F - \varepsilon_L)}\right]\rho \tag{33a}$$

$$\lfloor v \rfloor = v_e \frac{t - t_L}{t_F - t_L}(1 - \beta)(\varepsilon_F - \varepsilon_R) \tag{33b}$$

$$\lfloor \sigma \rfloor = E\frac{t - t_L}{t_F - t_L}[\beta(\varepsilon_F - \varepsilon_L) - (\varepsilon_L - \varepsilon_R)]. \tag{33c}$$

As can be seen, $\lfloor \rho \rfloor = \lfloor v \rfloor = \lfloor \sigma \rfloor = 0$ at $t = t_L$, and these jumps at $t = t_F$ obtain the following values:

$$\lfloor \rho \rfloor = \left[1 - \frac{1 - (2\varepsilon_F - \varepsilon_L - \varepsilon_R)}{1 - (\varepsilon_F - \varepsilon_L)}\right]\rho \tag{34a}$$

$$\lfloor v \rfloor = v_e(1 - \beta)(\varepsilon_F - \varepsilon_R) \tag{34b}$$

$$\lfloor \sigma \rfloor = E[\beta(\varepsilon_F - \varepsilon_L) - (\varepsilon_L - \varepsilon_R)] \tag{34c}$$

which exhibit the features similar to the dynamic case, eqn (28), except that the external forcing function does not play any role here. The difference between the analytical solutions for dynamic and static bars illustrates the fact that the post-limit response can be experimentally observed for a dynamic external forcing function applied at $x = L$ but not for a static one.

The analytical solutions presented here replicate the essential feature of localization problems, i.e., degradation of material properties with evolving localized deformations. They are comparable with the solutions obtained via other approaches (Chen, 1993a,b; Schreyer, 1990; Schreyer and Chen, 1986), except the jump in stress field. This jump in stress might be a macroscopic explanation of the driving force for a nonzero v_b. Since the final size of a localization zone and the post-limit load–displacement curve are frequently available, the only two parameters associated with the evolution of localization, t_F and ε_F or β, can be determined through simple bar tests.

3. DISCUSSION

Based on the essential features of localization phenomena, a partitioned-modeling approach has been used to predict the evolution of localization via moving jump forms of conservation laws in mass density and linear momentum. For the problems considered here, the conservation in density and momentum implies that in energy. To illustrate the procedure, analytical solutions have been given for one-dimensional dynamic and static problems. Although the uniqueness and stability as required by well-posedness have not been established on a sound mathematical foundation, the existence of a meaningful solution is guaranteed if a moving boundary between two domains of different constitutive relations is defined. One domain involves material of original properties, and the other is described by a degradation relation without unloading. The speed of the moving boundary is determined through resulting jumps in field variables. Because the evolution of localized deformations can also be predicted by those models of higher order terms in space and time, there must be a relationship between the speed of the moving boundary and the dispersion property of waves in higher order modes. A further study should be able to show this relationship, which might provide an alternative means to determine the parameters in the enhanced continua description.

Several assumptions, such as a boundary moving at constant speed, and small deformations with a constant strain field inside the localization zone, have been made to obtain closed-form solutions. Although these assumptions might not be valid in many cases, a heuristic path has been developed via one-dimensional analytical solutions towards the generalization of the proposed approach since the jump conditions can also be defined in higher dimensions. Also, other solutions may be possible if these assumptions are modified. In general, numerical methods must be invoked to solve the differential equations governing the partitioned-domains for given data. The solutions of field variables in conjunction with jump forms of conservation laws can be used to determine the configuration of the resulting solution domains via either an analytical or a numerical approach, depending on the specific case considered. To trace the history-dependent material flow with phase changes, a particle scheme (Sulsky *et al.*, 1994) might provide an efficient semi-discretization procedure in space for the partitioned-modeling approach via a moving boundary. As a result, a few integration points might be enough to resolve localized large deformations within a small zone, and a complete failure process, involving smeared and discrete mechanisms (Xie *et al.*, 1994), might be predicted for the safety assessment of large scale structures.

Acknowledgements This work was partially supported by the United States Department of Energy through Sandia National Laboratories (SNL) under Contract DE-AC04-94AL85000, and New Mexico Waste-management Education and Research Consortium (WERC). The interest shown by Dr M. Marietta of SNL is gratefully acknowledged. The authors are also grateful to Prof. Schreyer and Mr. Zuo for valuable joint discussions, and to reviewers for discerning comments on the paper.

REFERENCES

Achenbach, J. D. (1984). *Wave Propagation in Elastic Solids*. Elsevier Science Publishing Company, Inc., New York, NY, U.S.A.

Bazant, Z. P. and Belytschko, T. B. (1985). Wave propagation in a strain-softening bar: Exact solution. *J. Engng Mech.* **111**, 381–389.

Bazant, Z. P., Belytschko, T. B. and Chang, T. (1984). Continuum theory for strain softening. *J. Engng Mech.* **110**, 1666–1692.

Chen, Z. (1993a). A partitioned-solution method with moving boundaries for nonlocal plasticity. In *Modern Approaches to Plasticity* (Edited by D. Kolymbas), pp. 449–468. Elsevier, New York, NY, U.S.A.

Chen, Z. (1993b). A semi-analytical solution procedure for predicting damage evolution at interfaces. *Int. J. Num. Analyt. Meth. Geomech.* **17**, 807–819.

Chen, Z. and Clark, T. (1994). Some remarks on domain-transition problems. Submitted to *Archives of Mechanics*.

Chen, Z. and Schreyer, H. L. (1987). Simulation of soil–concrete interfaces with nonlocal constitutive models. *J. Engng Mech.* **113**, 1665–1677.

Chen, Z. and Schreyer, H. L. (1991). Secant structural solution strategies under element constraint for incremental damage. *Comput. Meth. Appl. Mech. Engng* **90**, 869–884.

Chen, Z. and Schreyer, H. L. (1994). On nonlocal damage models for interface problems. *Int. J. Solids Structures* **31**, 1241–1261.

Chen, Z. and Wang, M. L. (1993). A partitioned-solution method with moving boundaries for nonlocal creep damage of concrete. In *Creep and Shrinkage of Concrete* (Edited by Z. P. Bazant and I. Carol), pp. 393–398. E & FN Spon (An Imprint of Chapman & Hall). New York, NY, U.S.A.

Chorin, A. J. and Marsden, J. E. (1979). *A Mathematical Introduction to Fluid Mechanics*. Springer-Verlag, New York, NY, U.S.A.

De Borst, R., Sluys, L. J., Mühlhaus, H.-B and Pamin, J. (1993). Fundamental issues in finite element analyses of localization of deformation. *Engng Comput.* **10,** 99–121.

John, F. (1982). *Partial Differential Equations*. Springer-Verlag, New York, NY, U.S.A.

van Mier, J. G. M. (1984). Strain-softening of concrete under multiaxial loading conditions. Ph.D. dissertation, University of Eindhoven, The Netherlands.

Mühlhaus, H.-B. and Aifantis, E. C. (1991). A variational principle for gradient plasticity. *Int. J. Solids Structures* **28,** 845–857.

Neilsen, M. K. and Schreyer, H. L. (1993). Bifurcations in elastic–plastic materials. *Int. J. Solids Structures* **30,** 521–544.

Pietruszczak, St. and Mroz, A. (1981). Finite element analysis of deformation of strain-softening materials. *Int. J. Num. Meth. Engng* **17,** 327–334.

Pietruszczak, S. and Niu, X. (1993). On the description of localized deformation. *Int. J. Num. Analyt. Meth. Geomech.* **17,** 791–805.

Pijaudier-Cabot, G. and Bazant, Z. P. (1987). Nonlocal damage theory. *J. Engng Mech.* **113,** 1512–1533.

Pijaudier-Cabot, G. and Benallal, A. (1993). Strain localization and bifurcation in a nonlocal continuum. *Int. J. Solids Structures* **30,** 1761–1775.

Schreyer, H. L. (1990). Analytical solutions for nonlinear strain-gradient softening and localization. *J. Appl. Mech.* **57,** 522–528.

Schreyer, H. L. and Chen, Z. (1986). One dimensional softening with localization. *J. Appl. Mech.* **53,** 791–797.

Shah, S. P. and Gopalaratnam, V. S. (1985). Softening response of plain concrete in direct tension. *J. Am. Concr. Inst.* **82,** 310–323.

Sulsky, D., Chen, Z. and Schreyer, H. L. (1994). A particle method for history-dependent materials. *Comput. Meth. Appl. Mech. Engng* **118,** 179–196.

Triantafyllidis, N. and Aifantis, E. C. (1986). A gradient approach to localization of deformation: 1. Hyperelastic materials. *J. Elast.* **16,** 225–237.

Vardoulakis, I., Shah, K. R. and Papanastasiou, P. (1992). Modelling of tool–rock shear interfaces using gradient-dependent flow theory of plasticity. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **29,** 573–582.

Xie, M., Gerstle, W. and Chen, Z. (1994). Finite element analysis of combined smeared and discrete mechanisms in rock salt. In *Computer Methods and Advances in Geomechanics* (Edited by H. J. Siriwardane and M. M. Zaman), pp. 1659–1664. A. A. Balkema Publishers, Brookfield, VT, U.S.A.

Zbib, H. M. and Jubran, J. S. (1992). Dynamic shear banding: A three-dimensional analysis. *Int. J. Plast.* **8,** 619–641.